Lecture 02

Theory of Production

Varian (third edition, 1992)

Chambers (1988)

Heathfield & Wibe (1987)

On Production Theory

"I know that even as a student I was drawn to the theory of production rather than to the formally almost identical theory of consumer choice. It seemed more down to earth."

(*Robert M. Solow, "Growth Theory and After". Nobel Lecture, Stockholm, Sweden, December 8, 1987.*)

In comparison with other fields

- Game theory
- Economics of Information
- Microeconometrics
- Psychological & Experimental Economics
- Time Series Analysis

Neo-classical Theory of Production

- **Production function approach**
  - Production function → Profit max/Cost min → factor demands → Profit/Cost function
- **Dual approach to production analysis**
  - Profit/Cost function → factor demands → production function/structure

Building Blocks of the Theory

- Production function
- Cost minimization
- Profit maximization
- Cost function
- Profit function

Some basic concepts

- The set of all technologically feasible production plans is called the firm's **production possibilities set** and will be denoted by Y, a subset of R^n. The set Y is supposed to describe all patterns of inputs and outputs that are technologically feasible.

Input requirement set

- If **x** is a vector of inputs that can produce y units of output, then the **input requirement set** can be written as:

$$V(y) = \{\mathbf{x} \text{ in } R^n_+ : (y, -\mathbf{x}) \text{ is in } Y\}$$

The input requirement set is the set of all input combinations that produce at least y units of output.

1. Production function defined

- A production function gives the maximum possible output which can be produced from given quantities of a set of inputs. It can be denoted as :

$$y=f(\mathbf{x})$$

where **x** is an n-dimensional vector of nonnegative inputs and y is a scalor of nonnegative output.

Assumptions on production function

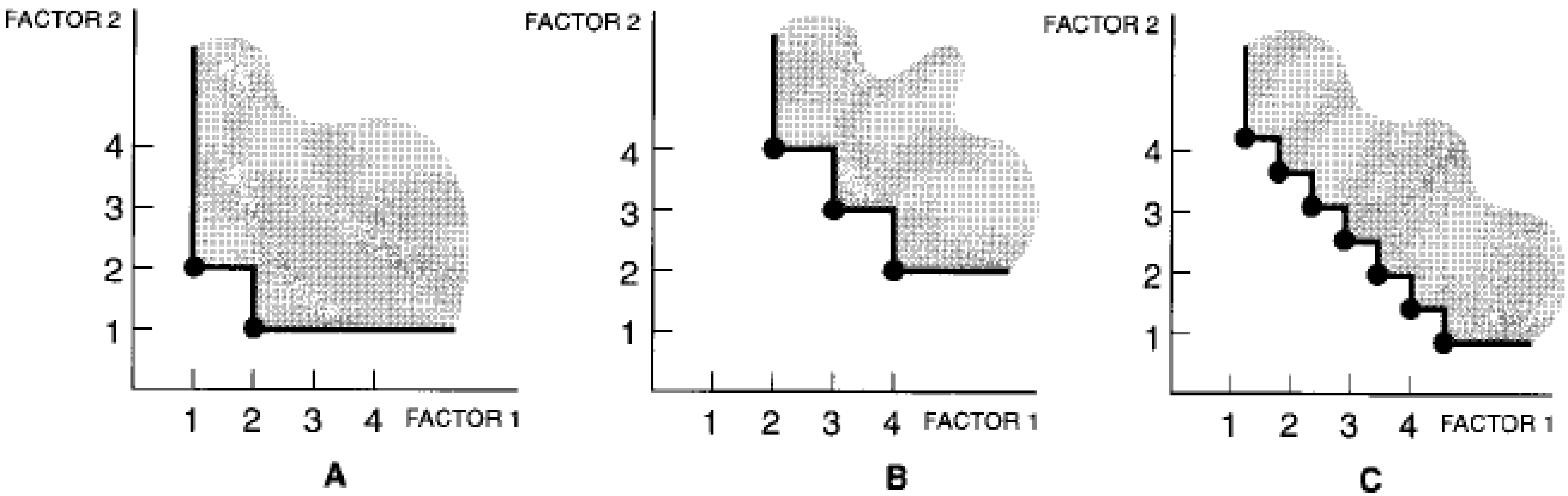
Properties of $f(x)$ (1A):

1. (a) if $x' \geq x$, then $f(x') \geq f(x)$ (monotonicity);
   (b) if $x' > x$, then $f(x') > f(x)$ (strict monotonicity)[2];
2. (a) $V(y) = \{x: f(x) \geq y\}$ is a convex set (quasi-concavity);
   (b) $f(\theta x^0 + (1-\theta)x^*) \geq \theta f(x^0) + (1-\theta) f(x^*)$ for $0 \leq \theta \leq 1$ (concavity);
3. (a) $f(0_n) = 0$, where 0_n is the null vector (weak essentiality);
   (b) $f(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) = 0$ for all x_i (strict essentiality);
4. the set $V(y)$ is closed and nonempty for all $y > 0$;
5. $f(x)$ is finite, nonnegative, real valued, and single valued for all nonnegative and finite x;
6. (a) $f(x)$ is everywhere continuous; and
   (b) $f(x)$ is everywhere twice-continuously differentiable.

Monotonicity

MONOTONICITY. *If* $\mathbf{x}$ *is in* $V(y)$ *and* $\mathbf{x}' \geq \mathbf{x}$, *then* $\mathbf{x}'$ *is in* $V(y)$.

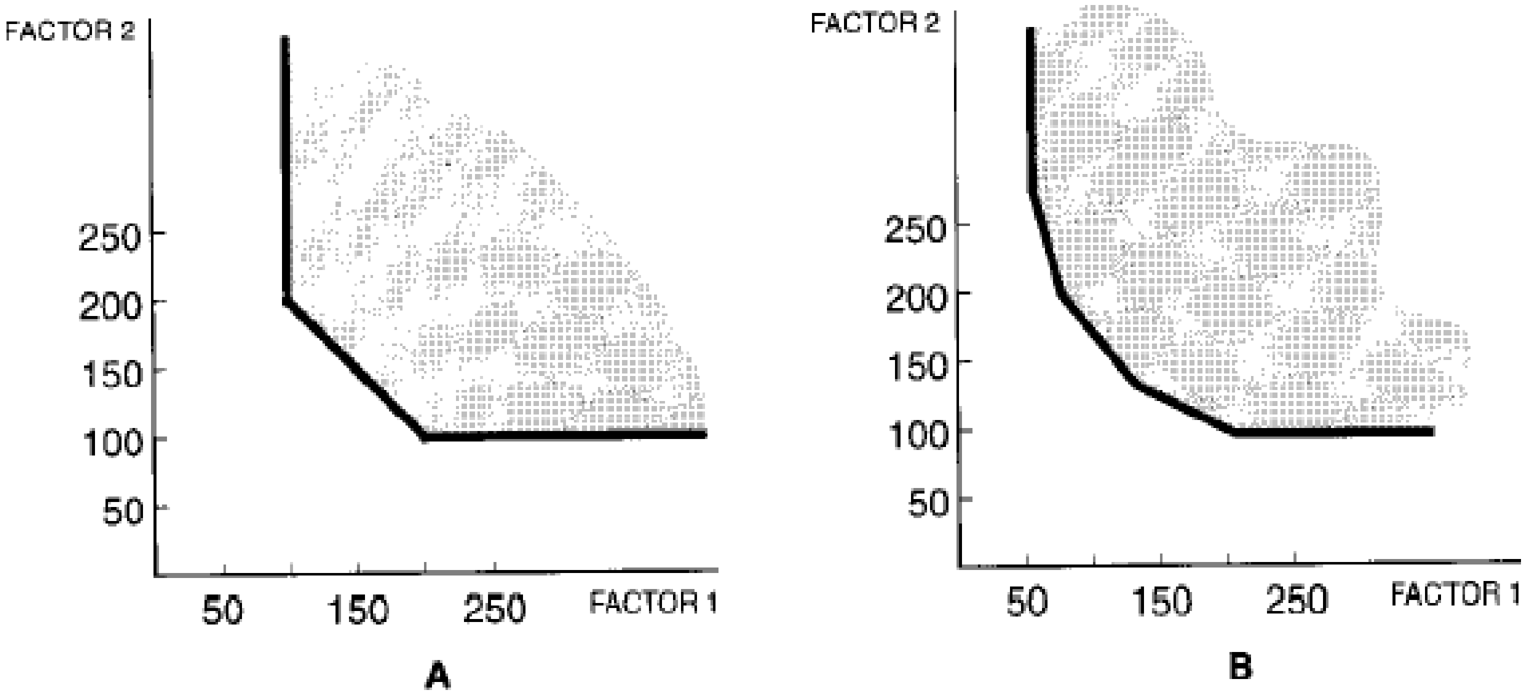
If we assume monotonicity, then the input requirement sets depicted in Figure 1.2 become the sets depicted in Figure 1.3.



Monotonicity. Here are the same three input requirement sets if we also assume monotonicity.

Convexity

CONVEXITY. *If* $\mathbf{x}$ *and* $\mathbf{x}'$ *are in* $V(y)$, *then* $t\mathbf{x} + (1-t)\mathbf{x}'$ *is in* $V(y)$ *for all* $0 \le t \le 1$. *That is,* $V(y)$ *is a* **convex set.**



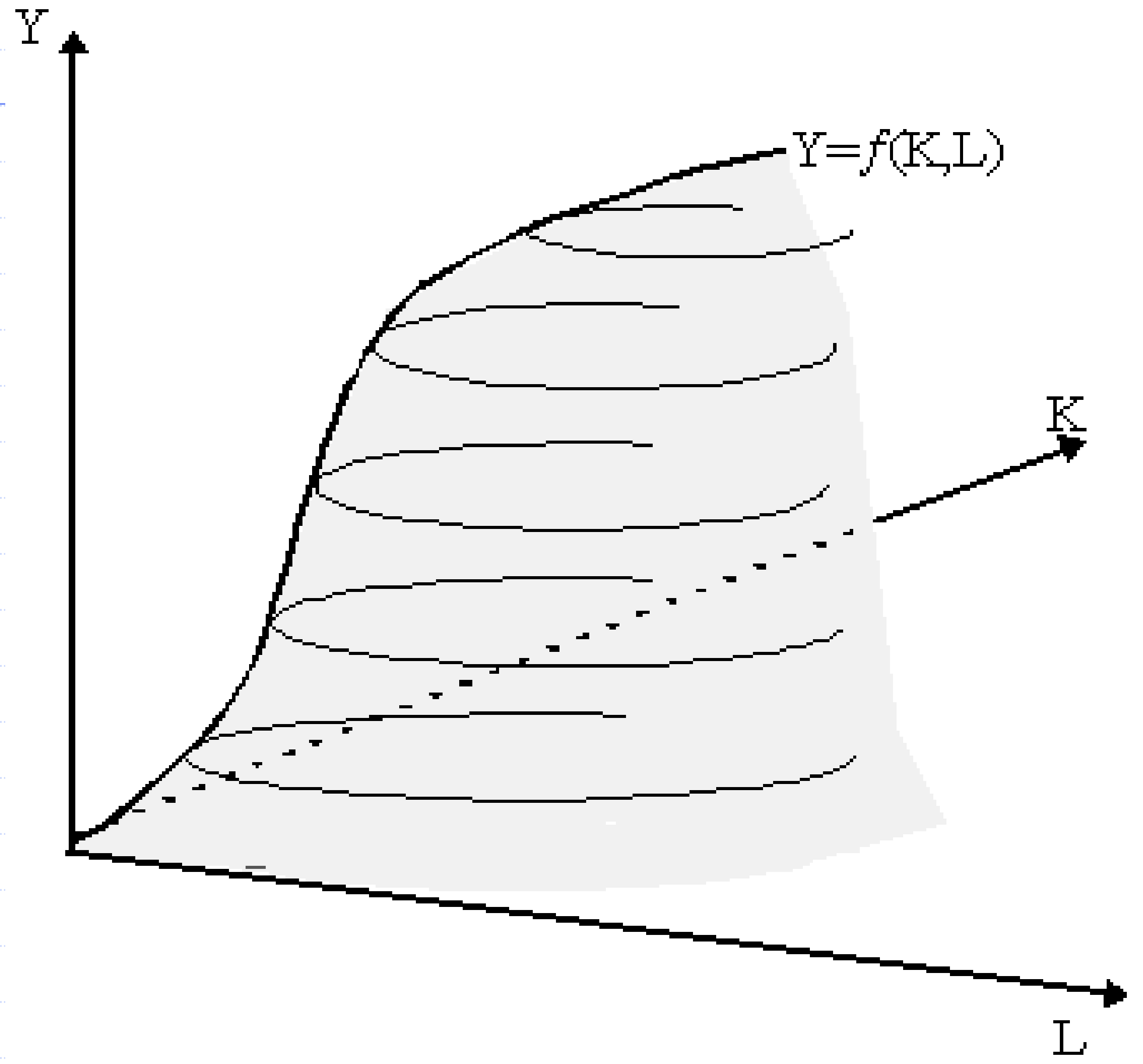
Regularity

REGULAR. *$V(y)$ is a closed, nonempty set for all $y \geq 0$.*

1. The nonemptiness implies that it is always possible to produce any positive output, i.e., a feasibility assumption.

2. The closedness assumption is made to include the boundary of V(y).

Production Function (figure 1)



Production function (figure 2)

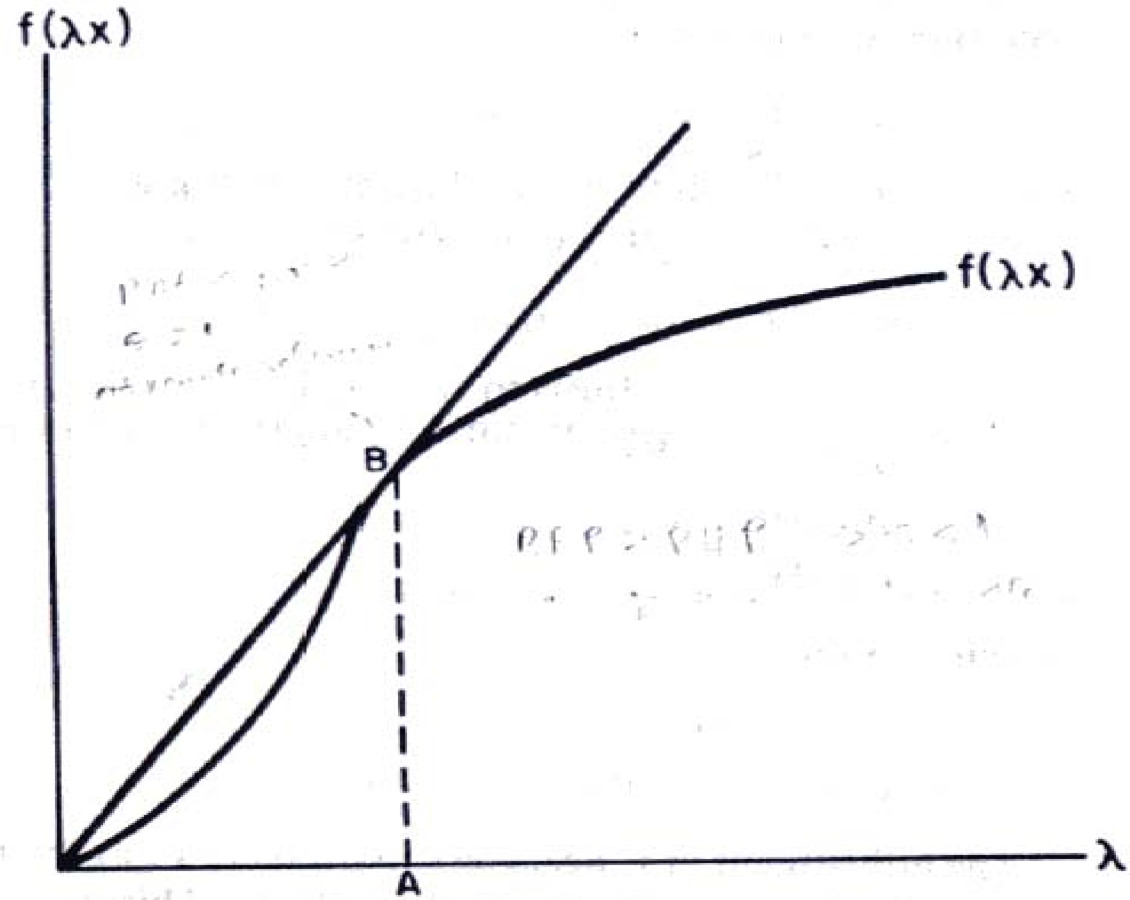


Figure 1.9 Law of variable proportions and elasticity of scale.

Characteristics of production structure

- Elastcity of substitution
- Elastsicity of scale
- Technical change

Technical rate of substitution

For a production function with two inputs $y=f(x_1,x_2)$

The associated change in the output is approximated by

$$dy = \frac{\partial f}{\partial x_1}\,dx_1 + \frac{\partial f}{\partial x_2}\,dx_2.$$

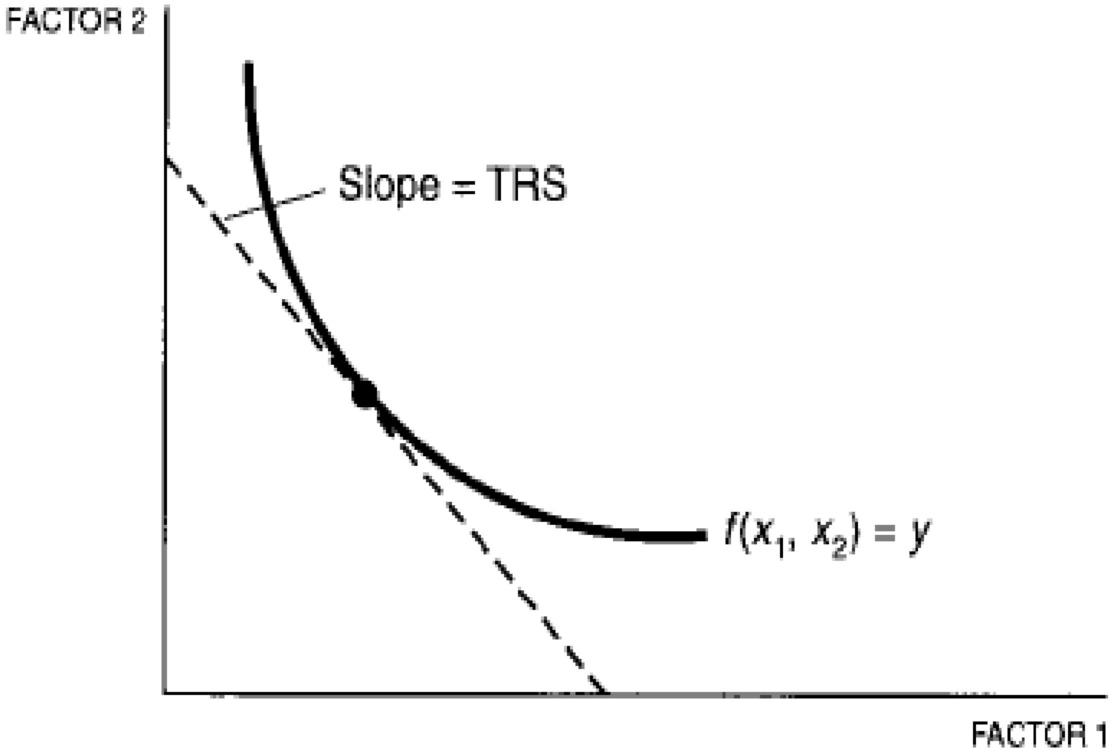
Since output remains constant, we have

$$0 = \frac{\partial f}{\partial x_1}\,dx_1 + \frac{\partial f}{\partial x_2}\,dx_2,$$

which can be solved for

$$\text{TRS} = \frac{dx_2}{dx_1} = -\frac{\partial f/\partial x_1}{\partial f/\partial x_2}.$$

Technical rate of substitution (graph)



The technical rate of substitution. The technical rate of substitution measures how one of the inputs must adjust in order to keep output constant when another input changes.

Elasticity of substitution (two inputs)

The technical rate of substitution measures the slope of an Isoquent. The elasticity of subsititution measures the curvature of an isoquant (Hicks, 1963)

$$\sigma = \frac{\frac{\Delta(x_2/x_1)}{x_2/x_1}}{\frac{\Delta TRS}{TRS}}$$

The elastcity of substitution measures the percentage change in the factor ratio diveided by the percentage change in the TRS, with output being held fixed.

Alternative expressions

- In logarithmic derivative form

$$\sigma = \frac{d \ln(x_1/x_2)}{d \ln |TRS|}$$

- In terms of derivatives of $f(x_1, x_2)$

$$\sigma = \frac{-f_1 f_2 (x_1 f_1 + x_2 f_2)}{x_1 x_2 (f_{11} f_2^2 - 2 f_{12} f_1 f_2 + f_{22} f_1^2)},$$

where $f_i \equiv \partial f / \partial x_i$ and $f_{ij} \equiv \partial^2 f / \partial x_i \partial x_j$.

Elasticity of substitution (n inputs)

- The Allen partial elastcity of substitution:

$$\sigma_{ij} = \frac{\sum_i x_i f_i}{x_i x_j} \frac{F_{ji}}{F}, \tag{1.15}$$

where F is the bordered Hessian determinant

$$F = \begin{vmatrix} 0 & f_1 & f_2 & \cdots & f_n \\ f_1 & f_{11} & f_{12} & \cdots & f_{nn} \\ f_2 & \cdots & \cdots & & f_{nn} \\ \vdots & \vdots & \vdots & & \vdots \\ f_n & f_{1n} & \cdots & \cdots & f_{nn} \end{vmatrix},$$

and F_{ij} is the cofactor associated with f_{ij}. Both σ_{ij}^D and σ_{ij} are symmetric measures of the degree of substitutability between two inputs.

Returns to scale

CONSTANT RETURNS TO SCALE. *A technology exhibits* **constant returns to scale** *if any of the following are satisfied:*

(3) $f(t\mathbf{x}) = tf(\mathbf{x})$ *for all* $t \geq 0$*; i.e., the production function* $f(\mathbf{x})$ *is homogeneous of degree 1.*

INCREASING RETURNS TO SCALE. *A technology exhibits* **increasing returns to scale** *if* $f(t\mathbf{x}) > tf(\mathbf{x})$ *for all* $t > 1$.

DECREASING RETURNS TO SCALE. *A technology exhibits* **decreasing returns to scale** *if* $f(t\mathbf{x}) < tf(\mathbf{x})$ *for all* $t > 1$.

Elasticity of Scale

Let $y = f(\mathbf{x})$ be the production function. Let t be a positive scalar, and consider the function $y(t) = f(t\mathbf{x})$. If $t = 1$, we have the current scale of operation; if $t > 1$, we are scaling all inputs up by t; and if $t < 1$, we are scaling all inputs down by t.

The elasticity of scale is given by

$$e(\mathbf{x}) = \frac{\frac{dy(t)}{y(t)}}{\frac{dt}{t}},$$

evaluated at $t = 1$. Rearranging this expression, we have

$$e(\mathbf{x}) = \frac{dy(t)}{dt}\frac{t}{y}\bigg|_{t=1} = \frac{df(t\mathbf{x})}{dt}\frac{t}{f(t\mathbf{x})}\bigg|_{t=1}.$$

Note that we must evaluate the expression at $t = 1$ to calculate the elasticity of scale at the point $\mathbf{x}$. We say that the technology exhibits locally increasing, constant, or decreasing returns to scale as $e(\mathbf{x})$ is greater, equal, or less than 1.

Alternative expression

For computational purposes, elasticity of scale can be directly evaluated to obtain

$$\left.\frac{\partial \ln f(\lambda x)}{\partial \ln \lambda}\right|_{\lambda=1} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{x_i}{y} = \sum_i \epsilon_i .$$

Thus, the elasticity of scale is the sum of the output elasticities.

2. Profit Maximization

If the firm produces only one output, the profit function can be written as

$$\pi(p, \mathbf{w}) = \max \; pf(\mathbf{x}) - \mathbf{wx}$$

where p is now the (scalar) price of output, $\mathbf{w}$ is the vector of factor prices, and the inputs are measured by the (nonnegative) vector $\mathbf{x} = (x_1, \ldots, x_n)$.

- First order condition

$$p\frac{\partial f(\mathbf{x}^*)}{\partial x_i} = w_i \qquad i = 1, \cdots, n.$$

Or

$$p\mathbf{D}f(\mathbf{x}^*) = \mathbf{w}.$$

$$\mathbf{D}f(\mathbf{x}^*) = \left(\frac{\partial f(\mathbf{x}^*)}{\partial x_1}, \ldots, \frac{\partial f(\mathbf{x}^*)}{\partial x_n}\right)$$

2.1 Comparative statics using the first-order conditions (single input)

The problem facing the firm is

$$\max_{x} \; pf(x) - wx.$$

first-order and second-order conditions

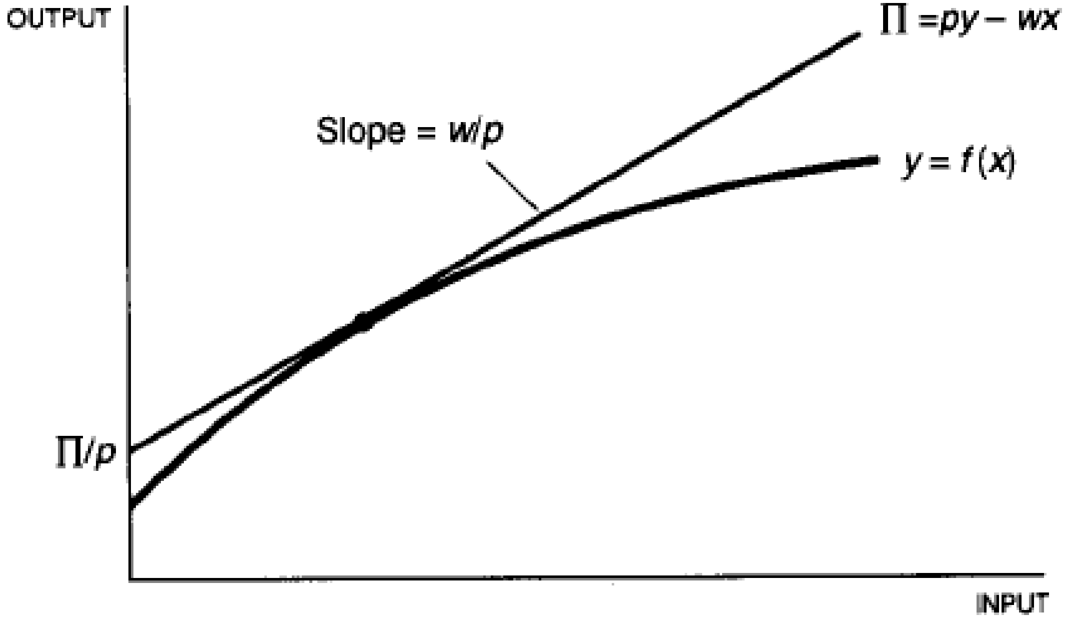
$$pf'(x(p,w)) - w \equiv 0$$
$$pf''(x(p,w)) \leq 0.$$

Differenciate the first-order condition w.r.t. w:

$$pf''(x(p,w))\frac{dx(p,w)}{dw} - 1 \equiv 0.$$

$$\frac{dx(p,w)}{dw} \equiv \frac{1}{pf''(x(p,w))}.$$

Second order condition (single input)



Profit maximization. The profit-maximizing amount of input occurs where the slope of the isoprofit line equals the slope of the production function.

$$\frac{d^2 f(x^*)}{dx^2} \leq 0.$$

Second order condition (multiple inputs)

matrix of second derivatives of the production function must be **negative semidefinite** at the optimal point; that is, the second-order condition requires that the Hessian matrix

$$\mathbf{D}^2 f(\mathbf{x}^*) = \left(\frac{\partial^2 f(\mathbf{x}^*)}{\partial x_i \partial x_j} \right)$$

must satisfy the condition that $\mathbf{h}\mathbf{D}^2 f(\mathbf{x}^*)\mathbf{h}^t \leq 0$ for all vectors $\mathbf{h}$. (The superscript t indicates the transpose operation.) Note that if there is only a single input, the Hessian matrix is a scalar and this condition reduces to the second-order condition we examined earlier for the single-input case.

Difficulties

- The production function may not be differentiable (e.g., Leontief case).
- Valid only for interior solutions. For boundary solusions, use Kuhn-Tucker conditions:

$$p\frac{\partial f(\mathbf{x})}{\partial x_i} - w_i \leq 0 \quad \text{if } x_i = 0$$

$$p\frac{\partial f(\mathbf{x})}{\partial x_i} - w_i = 0 \quad \text{if } x_i > 0.$$

- Profit can be unbounded (e.g., CRS)

$$pf(\mathbf{x}^*) - \mathbf{w}\mathbf{x}^* = \pi^* > 0.$$

$$pf(t\mathbf{x}^*) - \mathbf{w}t\mathbf{x}^* = t[pf(\mathbf{x}^*) - \mathbf{w}\mathbf{x}^*] = t\pi^* > \pi^*.$$

2.1 Comparative statics using the first- order conditions (two inputs, p=1)

$$\frac{\partial f(x_1(w_1,w_2),x_2(w_1,w_2))}{\partial x_1} \equiv w_1$$
$$\frac{\partial f(x_1(w_1,w_2),x_2(w_1,w_2))}{\partial x_2} \equiv w_2.$$

Differentiating with respect to w_1, we have

$$f_{11}\frac{\partial x_1}{\partial w_1} + f_{12}\frac{\partial x_2}{\partial w_1} = 1$$
$$f_{21}\frac{\partial x_1}{\partial w_1} + f_{22}\frac{\partial x_2}{\partial w_1} = 0.$$

Differentiating with respect to w_2, we have

$$f_{11}\frac{\partial x_1}{\partial w_2} + f_{12}\frac{\partial x_2}{\partial w_2} = 0$$
$$f_{21}\frac{\partial x_1}{\partial w_2} + f_{22}\frac{\partial x_2}{\partial w_2} = 1.$$

Comparative statics (two inputs, p=1)

Writing these equations in matrix form yields

$$\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \begin{pmatrix} \frac{\partial x_1}{\partial w_1} & \frac{\partial x_1}{\partial w_2} \\ \frac{\partial x_2}{\partial w_1} & \frac{\partial x_2}{\partial w_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Solving for the matrix of first derivatives, we have

$$\begin{pmatrix} \frac{\partial x_1}{\partial w_1} & \frac{\partial x_1}{\partial w_2} \\ \frac{\partial x_2}{\partial w_1} & \frac{\partial x_2}{\partial w_2} \end{pmatrix} = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}^{-1}.$$

The inverse of a symmetric negative definite matrix is symmetric negative definite.

1) $\partial x_i / \partial w_i < 0$, for $i = 1, 2$, since the diagonal entries of a negative definite matrix must be negative.

2) $\partial x_i / \partial w_j = \partial x_j / \partial w_i$ by the symmetry of the matrix.

Comparative statics (n inputs, p=1)

the first-order conditions for profit maximization

$$\mathbf{D}f(\mathbf{x}(\mathbf{w})) - \mathbf{w} \equiv \mathbf{0}.$$

If we differentiate with respect to $\mathbf{w}$, we get

$$\mathbf{D}^2 f(\mathbf{x}(\mathbf{w}))\mathbf{D}\mathbf{x}(\mathbf{w}) - \mathbf{I} \equiv \mathbf{0}.$$

Solving this equation for the substitution matrix, we find

$$\mathbf{D}\mathbf{x}(\mathbf{w}) \equiv [\mathbf{D}^2 f(\mathbf{x}(\mathbf{w}))]^{-1}.$$

$$\mathbf{dx} = \mathbf{Dx}(\mathbf{w})\mathbf{dw}^t$$

Multiplying both sides of this equation by $\mathbf{dw}$ yields

$$\mathbf{dw}\,\mathbf{dx} = \mathbf{dw}\mathbf{Dx}(\mathbf{w})\mathbf{dw}^t \leq 0.$$

2.2 Properties of the Profit Function

Properties of the profit function.

1) Nondecreasing in output prices, nonincreasing in input prices. If $p_i' \geq p_i$ *for all outputs and* $p_j' \leq p_j$ *for all inputs, then* $\pi(\mathbf{p}') \geq \pi(\mathbf{p})$.

2) Homogeneous of degree 1 in $\mathbf{p}$. $\pi(t\mathbf{p}) = t\pi(\mathbf{p})$ *for all* $t \geq 0$.

3) Convex in $\mathbf{p}$. *Let* $\mathbf{p}'' = t\mathbf{p} + (1-t)\mathbf{p}'$ *for* $0 \leq t \leq 1$. *Then* $\pi(\mathbf{p}'') \leq t\pi(\mathbf{p}) + (1-t)\pi(\mathbf{p}')$.

4) Continuous in $\mathbf{p}$. *The function* $\pi(\mathbf{p})$ *is continuous, at least when* $\pi(\mathbf{p})$ *is well-defined and* $p_i > 0$ *for* $i = 1, \ldots, n$.

Hotelling's lemma

Hotelling's lemma. *(The derivative property) Let* $y_i(\mathbf{p})$ *be the firm's net supply function for good* i. *Then*

$$y_i(\mathbf{p}) = \frac{\partial \pi(\mathbf{p})}{\partial p_i} \qquad \text{for } i = 1, \ldots, n,$$

assuming that the derivative exists and that $p_i > 0$.

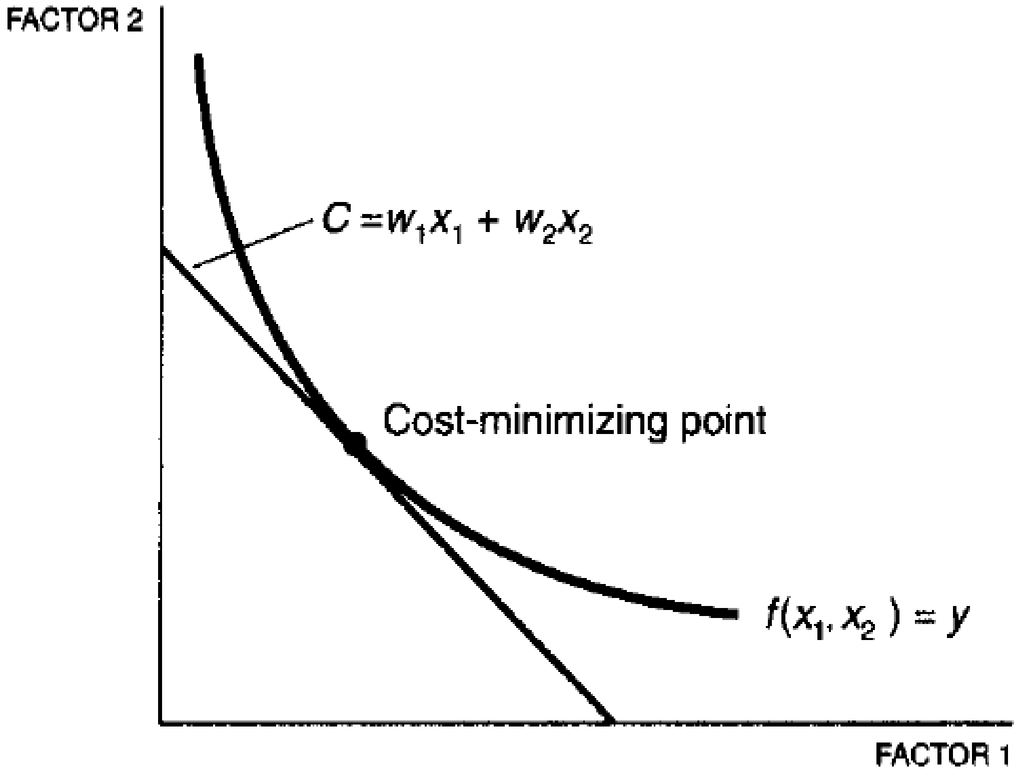
3. Cost minimization

$$\min_{\mathbf{x}} \mathbf{w}\mathbf{x}$$

$$\text{such that } f(\mathbf{x}) = y.$$

$$\mathcal{L}(\lambda, \mathbf{x}) = \mathbf{w}\mathbf{x} - \lambda(f(\mathbf{x}) - y)$$

Geometric intuition of cost min.



Cost minimization. At a point that minimizes costs, the isoquant must be tangent to the constant cost line.

Conditional factor demand functions

$\mathbf{x}(\mathbf{w}, y)$ must satisfy the first-order conditions

$$f(\mathbf{x}(\mathbf{w}, y)) \equiv y$$
$$\mathbf{w} - \lambda \mathbf{Df}(\mathbf{x}(\mathbf{w}, y)) \equiv \mathbf{0}.$$

we will consider a simple two-good example. In this case the first-order conditions look like

$$f(x_1(w_1, w_2, y), x_2(w_1, w_2, y)) \equiv y$$
$$w_1 - \lambda \frac{\partial f(x_1(w_1, w_2, y), x_2(w_1, w_2, y))}{\partial x_1} \equiv 0$$
$$w_2 - \lambda \frac{\partial f(x_1(w_1, w_2, y), x_2(w_1, w_2, y))}{\partial x_2} \equiv 0$$

Conditional factor demand (con.)

$$\frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial w_1}+\frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial w_1}\equiv 0$$

$$1-\lambda\left[\frac{\partial^2 f}{\partial x_1^2}\frac{\partial x_1}{\partial w_1}+\frac{\partial^2 f}{\partial x_1\partial x_2}\frac{\partial x_2}{\partial w_1}\right]-\frac{\partial f}{\partial x_1}\frac{\partial \lambda}{\partial w_1}\equiv 0$$

$$0-\lambda\left[\frac{\partial^2 f}{\partial x_2\partial x_1}\frac{\partial x_1}{\partial w_1}+\frac{\partial^2 f}{\partial x_2^2}\frac{\partial x_2}{\partial w_1}\right]-\frac{\partial f}{\partial x_2}\frac{\partial \lambda}{\partial w_1}\equiv 0.$$

These equations can be written in matrix form as

$$\begin{pmatrix} 0 & -f_1 & -f_2 \\ -f_1 & -\lambda f_{11} & -\lambda f_{21} \\ -f_2 & -\lambda f_{12} & -\lambda f_{22} \end{pmatrix}\begin{pmatrix} \frac{\partial \lambda}{\partial w_1} \\ \frac{\partial x_1}{\partial w_1} \\ \frac{\partial x_2}{\partial w_1} \end{pmatrix}\equiv\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}.$$

Using Cramer's rule

$$\frac{\partial x_1}{\partial w_1} = \frac{\begin{vmatrix} 0 & 0 & -f_2 \\ -f_1 & -1 & -\lambda f_{21} \\ -f_2 & 0 & -\lambda f_{22} \end{vmatrix}}{\begin{vmatrix} 0 & -f_1 & -f_2 \\ -f_1 & -\lambda f_{11} & -\lambda f_{21} \\ -f_2 & -\lambda f_{12} & -\lambda f_{22} \end{vmatrix}} = \frac{f_2^2}{H} < 0.$$

$$\frac{\partial x_2}{\partial w_1} = \frac{\begin{vmatrix} 0 & -f_1 & 0 \\ -f_1 & -\lambda f_{11} & 1 \\ -f_2 & -\lambda f_{12} & 0 \end{vmatrix}}{\begin{vmatrix} 0 & -f_1 & -f_2 \\ -f_1 & -\lambda f_{11} & -\lambda f_{21} \\ -f_2 & -\lambda f_{12} & -\lambda f_{22} \end{vmatrix}} = -\frac{f_2 f_1}{H} > 0.$$

3.1 Properties of the Cost Function

Properties of the cost function.

1) Nondecreasing in $\mathbf{w}$. *If* $\mathbf{w}' \geq \mathbf{w}$, *then* $c(\mathbf{w}', y) \geq c(\mathbf{w}, y)$.

2) Homogeneous of degree 1 in $\mathbf{w}$. $c(t\mathbf{w}, y) = tc(\mathbf{w}, y)$ *for* $t > 0$.

3) Concave in $\mathbf{w}$. $c(t\mathbf{w} + (1-t)\mathbf{w}', y) \geq tc(\mathbf{w}, y) + (1-t)c(\mathbf{w}', y)$ *for* $0 \leq t \leq 1$.

4) Continuous in $\mathbf{w}$. $c(\mathbf{w}, y)$ *is continuous as a function of* $\mathbf{w}$, *for* $\mathbf{w} \gg \mathbf{0}$.

Shephard's lemma

Shephard's lemma. *(The derivative property.) Let $x_i(\mathbf{w}, y)$ be the firm's conditional factor demand for input i. Then if the cost function is differentiable at $(\mathbf{w}, y)$, and $w_i > 0$ for $i = 1, \ldots, n$ then*

$$x_i(\mathbf{w}, y) = \frac{\partial c(\mathbf{w}, y)}{\partial w_i} \qquad i = 1, \cdots, n.$$

Proof. The proof is very similar to the proof of Hotelling's lemma. Let $\mathbf{x}^*$ be a cost-minimizing bundle that produces y at prices $\mathbf{w}^*$. Then define the function

$$g(\mathbf{w}) = c(\mathbf{w}, y) - \mathbf{w}\mathbf{x}^*.$$

Since $c(\mathbf{w}, y)$ is the cheapest way to produce y, this function is always nonpositive. At $\mathbf{w} = \mathbf{w}^*$, $g(\mathbf{w}^*) = 0$. Since this is a maximum value of $g(\mathbf{w})$, its derivative must vanish:

$$\frac{\partial g(\mathbf{w}^*)}{\partial w_i} = \frac{\partial c(\mathbf{w}^*, y)}{\partial w_i} - x_i^* = 0 \qquad i = 1, \cdots, n$$

Hence, the cost-minimizing input vector is just given by the vector of derivatives of the cost function with respect to the prices. ∎

4. Duality

- Given a cost function we can ”solve for” a technology that could have generated that cost functiton. This means that the cost function contains essentially the same information that the production function contains. Any concept defined in terms of the properties of the production function has a ”dual” definition in terms of the properties of the cost function and vice versa (Varian, 1992).

Sufficient conditions for cost functions

When $\phi(\mathbf{w}, y)$ is a cost function. *Let $\phi(\mathbf{w}, y)$ be a differentiable function satisfying*

1) $\phi(t\mathbf{w}, y) = t\phi(\mathbf{w}, y)$ for all $t \geq 0$;

2) $\phi(\mathbf{w}, y) \geq 0$ for $\mathbf{w} \geq 0$ and $y \geq 0$;

3) $\phi(\mathbf{w}', y) \geq \phi(\mathbf{w}, y)$ for $\mathbf{w}' \geq \mathbf{w}$;

4) $\phi(\mathbf{w}, y)$ is concave in $\mathbf{w}$.

Then $\phi(\mathbf{w}, y)$ is the cost function for the technology defined by $V^*(y) = \{\mathbf{x} \geq \mathbf{0} : \mathbf{w}\mathbf{x} \geq \phi(\mathbf{w}, y),\ \text{for all } \mathbf{w} \geq \mathbf{0}\}$.

Elasticity of scale and the cost function

Given a production function $f(\mathbf{x})$ we can consider the local measure of returns to scale known as the **elasticity of scale**:

$$e(\mathbf{x}) = \frac{df(t\mathbf{x})}{dt}\frac{t}{f(\mathbf{x})}\Big|_{t=1}$$

then

$$e(\mathbf{x}^*) = \frac{c(\mathbf{w},y)/y}{\partial c(\mathbf{w},y)/\partial y} = \frac{AC(y)}{MC(y)}$$

notice

$$e(\mathbf{x}^*) = \frac{\sum_{i=1}^n \frac{\partial f(\mathbf{x}^*)}{\partial x_i} x_i^*}{f(\mathbf{x}^*)}.$$

Since $\mathbf{x}^*$ minimizes costs it satisfies the first-order conditions that $w_i = \lambda \frac{\partial f(x^*)}{\partial x_i}$. Furthermore, by the envelope theorem, $\lambda = \partial c(\mathbf{w},y)/\partial y$. (See Chapter 5, page 76.) Thus,

$$e(\mathbf{x}^*) = \frac{\sum_{i=1}^n w_i x_i^*}{\lambda f(\mathbf{x}^*)} = \frac{c(\mathbf{w},y)/f(\mathbf{x}^*)}{\partial c(\mathbf{w},y)/\partial y} = \frac{AC(y)}{MC(y)}.$$

Elasticity of substitution and the cost function

- The (direct) elastcity of substitution:

$$\sigma \equiv \frac{d\ln(x_2/x_1)}{d\ln(f_1/f_2)} = \frac{d\ln(x_2/x_1)}{d\ln(w_1/w_2)}$$

The Allen partial elastcity of substitution:

$$\sigma_{ij} = \frac{F_{ji}}{F}\frac{\sum_{i=1}^{n}(\partial f(x)/\partial x_i)x_i}{x_i x_j} = \epsilon_{ij}/S_j$$

Here, S_j is the jth cost share $(w_j x_j / c(w, y))$.

Data for application

- Chinese aggregate economy
- Chinese provincial data
- Chinese enterprise data
- Pen world data?
- Summers & Heston (1996)
- Barro and Sala-I-Martin (1995)
- Nordic plant data